Sine/Cosine Expressions

Warmup

Simplify

1) $1 - \dfrac{\sin^2 x}{1 + \cos x}$ $\quad$ $\cos x$

2) $1 - \dfrac{\cos^2 x}{1 + \sin x}$ $\quad$ $\sin x$

3) $\dfrac{\sin x}{\cos x + 1} + \dfrac{\cos x - 1}{\sin x}$ $\quad$ 0

Chapter 5
Trigonometric Identities

1. **Trigonometric Identities**
2. Sum and Difference Identities
3. Double-Angle Identities
4. Half-Angle Identities
5. Product-to-Sum Identities

5.1 - Trigonometric Identities

Definitions

- **Identity**: an equation that is true for all values for which the equation is defined.

$$\tan x = \frac{\sin x}{\cos x} \qquad x \neq \frac{(2n+1)\pi}{2}$$

- **Conditional**: an equation that is true for some values.

$$\tan x = 0 \qquad x = n\pi$$

- **Contradiction**: an equation that has no true values.

$$\sin^2 x + \cos^2 x = 2$$

5.1 - Trigonometric Identities

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad x \neq n\pi$$

$$\sec x = \frac{1}{\cos x} \qquad x \neq \frac{(2n+1)\pi}{2}$$

$$\cot x = \frac{1}{\tan x} \qquad x \neq \frac{n\pi}{2}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x} \qquad x \neq \frac{(2n+1)\pi}{2}$$

$$\cot x = \frac{\cos x}{\sin x} \qquad x \neq n\pi$$

5.1 - Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$	all real numbers
$\tan^2 x + 1 = \sec^2 x$	$x \neq \frac{(2n+1)\pi}{2}$
$\cot^2 x + 1 = \csc^2 x$	$x \neq n\pi$

Even-Odd Identities

$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\csc(-x) = -\csc x$
	$\tan(-x) = -\tan x$
	$\cot(-x) = -\cot x$

5.1 - Trigonometric Identities

Simplify the expression

$a.\ \tan x \sin x + \cos x$ $\sec x$

$b.\ \dfrac{1}{\csc^2 x} + \dfrac{1}{\sec^2 x}$ 1

$c.\ \dfrac{1}{\cos^2 x} - 1$ $\tan^2 x$

5.1 - Trigonometric Identities

Verify the identity

$$(\sin x - \cos x)^2 - 1 = -2\sin x \cos x$$

5.1 - Trigonometric Identities

Verify the identity

$a.\ (1 + \sin x)[1 + \sin(-x)] = \cos^2 x$

$b.\ \dfrac{\tan x - \cot x}{\tan x + \cot x} = \sin^2 x - \cos^2 x$

$c.\ -\dfrac{\cot^2 x}{1 - \csc x} = \dfrac{\sin x + 1}{\sin x}$

Chapter 5
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5.2 - Sum and Difference Identities

Trigonometric functions are not distributive, for example:

$$\cos(A + B) \neq \cos A + \cos B$$

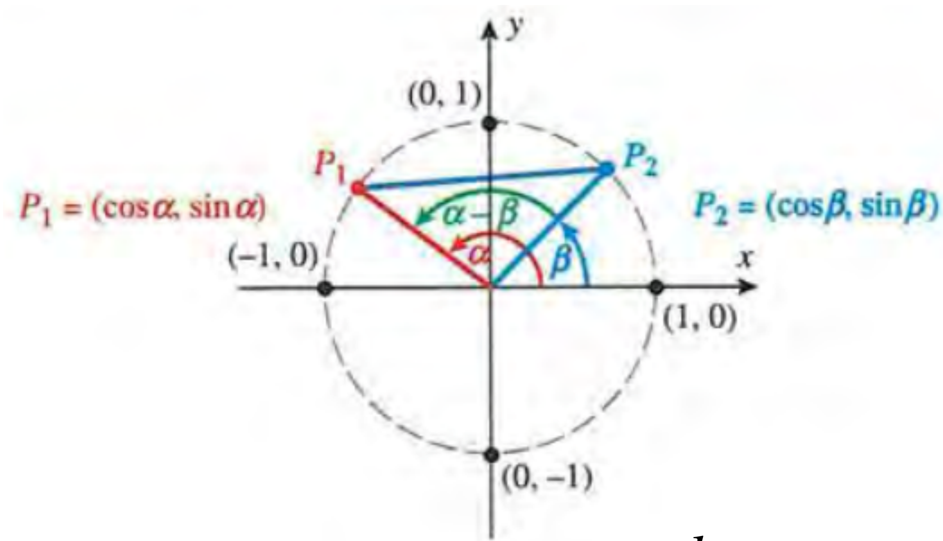
Easy to prove. Let $A = \pi$ and $B = 0$.

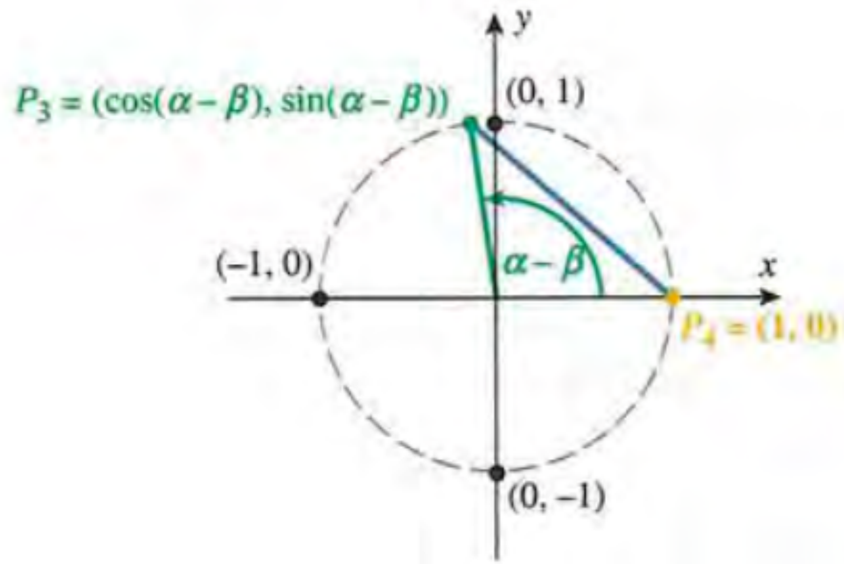
$$\cos(\pi + 0) = \cos \pi = -1$$

$$\cos \pi + \cos 0 = -1 + 1 = 0$$

5.2 - Sum and Difference Identities

Deriving the Cosine difference of angles





$$d_{P_1 \text{ to } P_2} = d_{P_3 \text{ to } P_4}$$

$P_1 = (\cos\alpha, \sin\alpha)$

$P_2 = (\cos\beta, \sin\beta)$

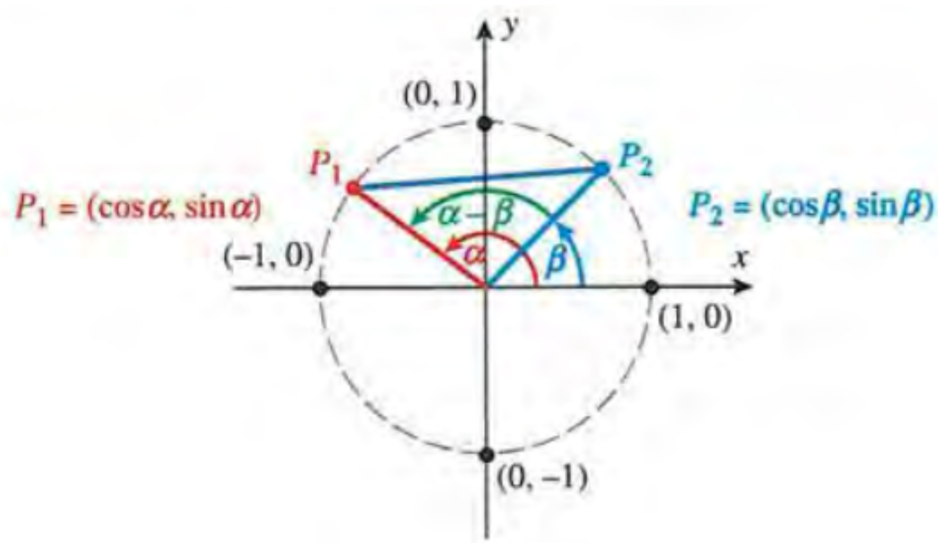
$P_3 = (\cos(\alpha-\beta), \sin(\alpha-\beta))$

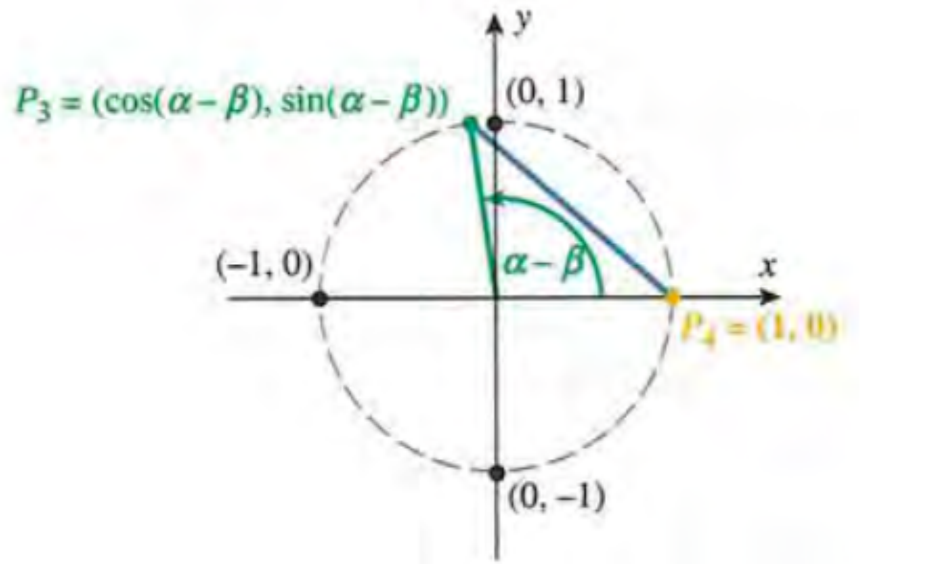
$P_4 = (1, 0)$

$$\sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2} = \sqrt{(1-\cos(\alpha-\beta))^2 + (0-\sin(\alpha-\beta))^2}$$

5.2 - Sum and Difference Identities

Deriving the Cosine difference of angles





$$\sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2} = \sqrt{(1 - \cos(\alpha-\beta))^2 + (0 - \sin(\alpha-\beta))^2}$$

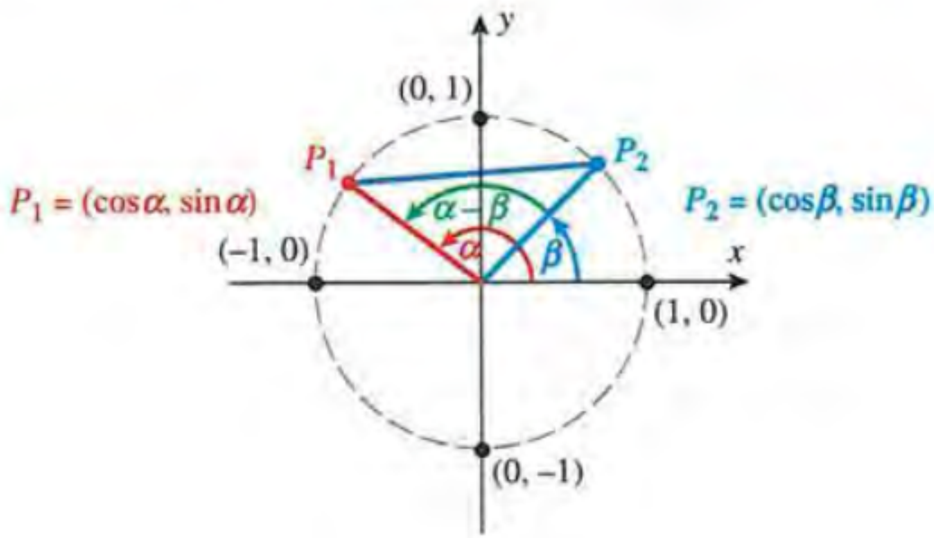
$$(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2 = (1 - \cos(\alpha-\beta))^2 + (0 - \sin(\alpha-\beta))^2$$

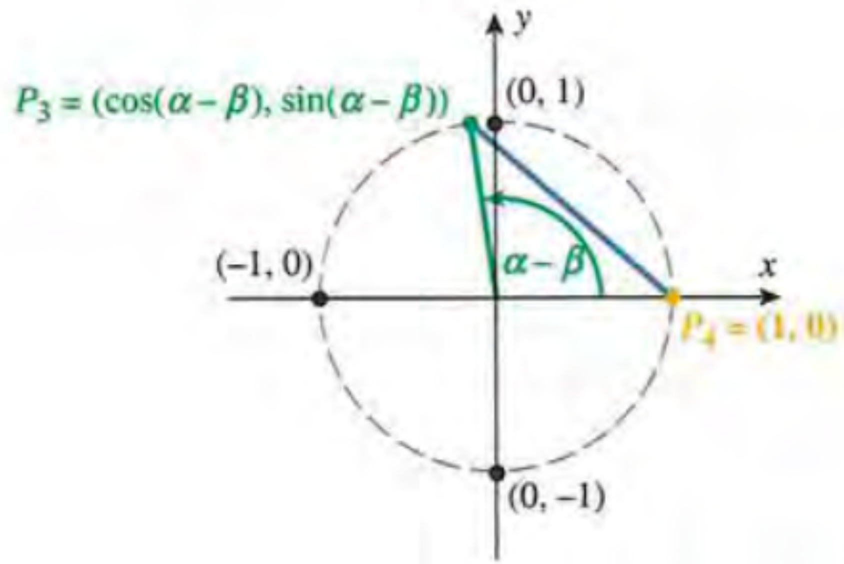
$$\cos^2\beta - 2\cos\alpha\cos\beta + \cos^2\alpha + \sin^2\beta - 2\sin\alpha\sin\beta + \sin^2\alpha$$
$$= 1 - 2\cos(\alpha-\beta) + \cos^2(\alpha-\beta) + \sin^2(\alpha-\beta)$$

$$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha-\beta)$$

5.2 - Sum and Difference Identities

Deriving the Cosine difference of angles





$$2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = 2 - 2\cos(\alpha - \beta)$$

$$-2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta = -2\cos(\alpha - \beta)$$

$$\boxed{\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)}$$

5.2 - Sum and Difference Identities

Deriving the Cosine sum of angles

Start with the cosine difference identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Substitute β with $-\beta$

$$\cos(\alpha - (-\beta)) = \cos\alpha\cos(-\beta) + \sin\alpha\sin(-\beta)$$

$$\boxed{\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

5.2 - Sum and Difference Identities

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Evaluate each of the following exactly:

$a.\ \cos 15^\circ$ $\qquad \cos(45^\circ - 30^\circ) = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

$b.\ \cos\left(\dfrac{7\pi}{12}\right)$ $\qquad \cos\left(\dfrac{4\pi}{12} + \dfrac{3\pi}{12}\right) = \dfrac{\sqrt{2}-\sqrt{6}}{4}$

$c.\ \cos 75^\circ$ $\qquad \cos(120^\circ - 45^\circ) = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

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